

# Transition maps between the 24 bases for a Leonard pair

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## Abstract

Let  $V$  denote a vector space with finite positive dimension. We consider a pair of linear transformations  $A : V \rightarrow V$  and  $A^* : V \rightarrow V$  that satisfy (i) and (ii) below:

- (i) There exists a basis for  $V$  with respect to which the matrix representing  $A$  is irreducible tridiagonal and the matrix representing  $A^*$  is diagonal.
- (ii) There exists a basis for  $V$  with respect to which the matrix representing  $A^*$  is irreducible tridiagonal and the matrix representing  $A$  is diagonal.

We call such a pair a *Leonard pair* on  $V$ . In an earlier paper we described 24 special bases for  $V$ . One feature of these bases is that with respect to each of them the matrices that represent  $A$  and  $A^*$  are (i) diagonal and irreducible tridiagonal or (ii) irreducible tridiagonal and diagonal or (iii) lower bidiagonal and upper bidiagonal or (iv) upper bidiagonal and lower bidiagonal. For each ordered pair of bases among the 24, there exists a unique linear transformation from  $V$  to  $V$  that sends the first basis to the second basis; we call this the transition map. In this paper we find each transition map explicitly as a polynomial in  $A, A^*$ .

## 1 Leonard pairs

We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix  $X$  is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume  $X$  is tridiagonal. Then  $X$  is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper  $\mathbb{K}$  will denote a field.

**Definition 1.1** [41] Let  $V$  denote a vector space over  $\mathbb{K}$  with finite positive dimension. By a *Leonard pair* on  $V$  we mean an ordered pair  $A, A^*$  where  $A : V \rightarrow V$  and  $A^* : V \rightarrow V$  are linear transformations that satisfy (i) and (ii) below:

- (i) There exists a basis for  $V$  with respect to which the matrix representing  $A$  is irreducible tridiagonal and the matrix representing  $A^*$  is diagonal.
- (ii) There exists a basis for  $V$  with respect to which the matrix representing  $A^*$  is irreducible tridiagonal and the matrix representing  $A$  is diagonal.

**Note 1.2** It is a common notational convention to use  $A^*$  to represent the conjugate-transpose of  $A$ . We are not using this convention. In a Leonard pair  $A, A^*$  the linear transformations  $A$  and  $A^*$  are arbitrary subject to (i) and (ii) above.

We refer the reader to [9, 11, 25, 28–34, 36, 39–41, 43–50, 52, 54, 55, 57] for background on Leonard pairs. We especially recommend the survey [50]. See [1–8, 10, 12–24, 26, 27, 35, 37, 38, 42, 51, 53, 56] for related topics.

## 2 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let  $d$  denote a nonnegative integer and let  $\text{Mat}_{d+1}(\mathbb{K})$  denote the  $\mathbb{K}$ -algebra consisting of all  $d+1$  by  $d+1$  matrices that have entries in  $\mathbb{K}$ . We index the rows and columns by  $0, 1, \dots, d$ . We let  $\mathbb{K}^{d+1}$  denote the  $\mathbb{K}$ -vector space of all  $d+1$  by 1 matrices that have entries in  $\mathbb{K}$ . We index the rows by  $0, 1, \dots, d$ . We view  $\mathbb{K}^{d+1}$  as a left module for  $\text{Mat}_{d+1}(\mathbb{K})$ . We observe this module is irreducible. For the rest of this paper, let  $\mathcal{A}$  denote a  $\mathbb{K}$ -algebra isomorphic to  $\text{Mat}_{d+1}(\mathbb{K})$  and let  $V$  denote an irreducible left  $\mathcal{A}$ -module. We remark that  $V$  is unique up to isomorphism of  $\mathcal{A}$ -modules, and that  $V$  has dimension  $d+1$ . By a *basis* for  $V$  we mean a sequence of vectors that are linear independent and span  $V$ . We emphasize that the ordering is important. Let  $\{v_i\}_{i=0}^d$  denote a basis for  $V$ . For  $X \in \mathcal{A}$  and  $Y \in \text{Mat}_{d+1}(\mathbb{K})$ , we say  $Y$  *represents*  $X$  with respect to  $\{v_i\}_{i=0}^d$  whenever  $Xv_j = \sum_{i=0}^d Y_{ij}v_i$  for  $0 \leq j \leq d$ . For  $A \in \mathcal{A}$  we say  $A$  is *multiplicity-free* whenever it has  $d+1$  mutually distinct eigenvalues in  $\mathbb{K}$ . Assume  $A$  is multiplicity-free. Let  $\{\theta_i\}_{i=0}^d$  denote an ordering of the eigenvalues of  $A$ , and for  $0 \leq i \leq d$  put

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \quad (1)$$

where  $I$  denotes the identity of  $\mathcal{A}$ . We observe (i)  $AE_i = \theta_i E_i$  ( $0 \leq i \leq d$ ); (ii)  $E_i E_j = \delta_{i,j} E_i$  ( $0 \leq i, j \leq d$ ); (iii)  $\sum_{i=0}^d E_i = I$ ; (iv)  $A = \sum_{i=0}^d \theta_i E_i$ . Let  $\mathcal{D}$  denote the subalgebra of  $\mathcal{A}$  generated by  $A$ . Using (i)–(iv) we find the sequence  $\{E_i\}_{i=0}^d$  is a basis for the  $\mathbb{K}$ -vector space  $\mathcal{D}$ . We call  $E_i$  the *primitive idempotent* of  $A$  associated with  $\theta_i$ . It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \quad (\text{direct sum}).$$

For  $0 \leq i \leq d$ ,  $E_i V$  is the (one dimensional) eigenspace of  $A$  in  $V$  associated with the eigenvalue  $\theta_i$ , and  $E_i$  acts on  $V$  as the projection onto this eigenspace.

By a *Leonard pair in  $\mathcal{A}$*  we mean an ordered pair of elements taken from  $\mathcal{A}$  that act on  $V$  as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

**Definition 2.1** [41] By a *Leonard system* in  $\mathcal{A}$  we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

- (i) Each of  $A, A^*$  is a multiplicity-free element in  $\mathcal{A}$ .
- (ii)  $\{E_i\}_{i=0}^d$  is an ordering of the primitive idempotents of  $A$ .
- (iii)  $\{E_i^*\}_{i=0}^d$  is an ordering of the primitive idempotents of  $A^*$ .
- (iv) For  $0 \leq i, j \leq d$ ,

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases} \quad (2)$$

- (v) For  $0 \leq i, j \leq d$ ,

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases} \quad (3)$$

Leonard systems are related to Leonard pairs as follows. Let  $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Then  $A, A^*$  is a Leonard pair in  $\mathcal{A}$  [49, Section 3]. Conversely, suppose  $A, A^*$  is a Leonard pair in  $\mathcal{A}$ . Then each of  $A, A^*$  is multiplicity-free [41, Lemma 1.3]. Moreover there exists an ordering  $\{E_i\}_{i=0}^d$  of the primitive idempotents of  $A$  and there exists an ordering  $\{E_i^*\}_{i=0}^d$  of the primitive idempotents of  $A^*$  such that  $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  is a Leonard system in  $\mathcal{A}$  [49, Lemma 3.3].

### 3 The 24 bases

Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . In [43] we obtained 24 special bases for  $V$ , on which  $A$  and  $A^*$  act in an attractive fashion. In this section we review these bases. First we recall some notation.

**Definition 3.1** Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . For  $0 \leq i \leq d$  we let  $\theta_i$  (resp.  $\theta_i^*$ ) denote the eigenvalue of  $A$  (resp.  $A^*$ ) associated with  $E_i$  (resp.  $E_i^*$ ). We refer to  $\{\theta_i\}_{i=0}^d$  (resp.  $\{\theta_i^*\}_{i=0}^d$ ) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of  $\Phi$ . We observe  $\{\theta_i\}_{i=0}^d$  (resp.  $\{\theta_i^*\}_{i=0}^d$ ) are mutually distinct and contained in  $\mathbb{K}$ .

For an indeterminate  $\lambda$  let  $\mathbb{K}[\lambda]$  denote the  $\mathbb{K}$ -algebra of all polynomials in  $\lambda$  that have coefficients in  $\mathbb{K}$ .

**Definition 3.2** Referring to Definition 3.1, for  $0 \leq i \leq d$  we define polynomials  $\tau_i$ ,  $\eta_i$ ,  $\tau_i^*$ ,  $\eta_i^*$  in  $\mathbb{K}[\lambda]$  as follows:

$$\begin{aligned}\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\ \eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\ \tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\ \eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).\end{aligned}$$

Note that each of  $\tau_i$ ,  $\eta_i$ ,  $\tau_i^*$ ,  $\eta_i^*$  is monic with degree  $i$ .

**Lemma 3.3** [43, Theorem 9.1] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\xi_0$ ,  $\xi_d$ ,  $\xi_0^*$ ,  $\xi_d^*$  denote nonzero vectors in  $V$  such that*

$$\xi_0 \in E_0V, \quad \xi_d \in E_dV, \quad \xi_0^* \in E_0^*V, \quad \xi_d^* \in E_d^*V. \quad (4)$$

*Then each of the 24 sequences below is a basis for  $V$ :*

$$\begin{aligned}& \{E_i \xi_0^*\}_{i=0}^d, & \{E_i \xi_d^*\}_{i=0}^d, & \{E_{d-i} \xi_0^*\}_{i=0}^d, & \{E_{d-i} \xi_d^*\}_{i=0}^d, \\& \{E_i^* \xi_0\}_{i=0}^d, & \{E_i^* \xi_d\}_{i=0}^d, & \{E_{d-i}^* \xi_0\}_{i=0}^d, & \{E_{d-i}^* \xi_d\}_{i=0}^d, \\& \{\tau_i(A) \xi_0^*\}_{i=0}^d, & \{\tau_i(A) \xi_d^*\}_{i=0}^d, & \{\eta_i(A) \xi_0^*\}_{i=0}^d, & \{\eta_i(A) \xi_d^*\}_{i=0}^d, \\& \{\tau_{d-i}^*(A^*) \xi_0\}_{i=0}^d, & \{\tau_{d-i}^*(A^*) \xi_d\}_{i=0}^d, & \{\eta_{d-i}^*(A^*) \xi_0\}_{i=0}^d, & \{\eta_{d-i}^*(A^*) \xi_d\}_{i=0}^d, \\& \{\tau_{d-i}(A) \xi_0^*\}_{i=0}^d, & \{\tau_{d-i}(A) \xi_d^*\}_{i=0}^d, & \{\eta_{d-i}(A) \xi_0^*\}_{i=0}^d, & \{\eta_{d-i}(A) \xi_d^*\}_{i=0}^d, \\& \{\tau_i^*(A^*) \xi_0\}_{i=0}^d, & \{\tau_i^*(A^*) \xi_d\}_{i=0}^d, & \{\eta_i^*(A^*) \xi_0\}_{i=0}^d, & \{\eta_i^*(A^*) \xi_d\}_{i=0}^d.\end{aligned}$$

**Note 3.4** Referring to Lemma 3.3 and with respect to each of the 24 bases, the matrices that represent  $A$  and  $A^*$  are (i) diagonal and irreducible tridiagonal (row 1); (ii) irreducible tridiagonal and diagonal (row 2); (iii) lower bidiagonal and upper bidiagonal (rows 3, 4); (iv) upper bidiagonal and lower bidiagonal (rows 5, 6). See [43] for more information about the 24 bases.

Referring to Lemma 3.3, for each ordered pair of bases among the 24 there exists a unique linear transformation from  $V$  to  $V$  that sends the first basis to the second basis; we call this the *transition map*. In this paper we find each transition map explicitly as a polynomial in  $A, A^*$ . Before we display the transition maps, we will review some basic facts and prove a few lemmas about Leonard pairs.

## 4 The $D_4$ action

Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Then each of the following is a Leonard system in  $\mathcal{A}$ :

$$\begin{aligned}\Phi^* &:= (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \\ \Phi^\downarrow &:= (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\Downarrow &:= (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).\end{aligned}$$

Viewing  $*$ ,  $\downarrow$ ,  $\Downarrow$  as permutations on the set of all the Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \quad (5)$$

$$\Downarrow* = *\downarrow, \quad \downarrow* = *\Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow. \quad (6)$$

The group generated by symbols  $*$ ,  $\downarrow$ ,  $\Downarrow$  subject to the relations (5), (6) is the dihedral group  $D_4$ . We recall  $D_4$  is the group of symmetries of a square, and has 8 elements. Apparently  $*$ ,  $\downarrow$ ,  $\Downarrow$  induce an action of  $D_4$  on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this  $D_4$  action. The relatives of  $\Phi$  are as follows:

name	relative
$\Phi$	$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi\downarrow$	$(A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
$\Phi\Downarrow$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi\downarrow\Downarrow$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
$\Phi^*$	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi\downarrow^*$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi\Downarrow^*$	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$
$\Phi\downarrow\Downarrow^*$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$

## 5 The parameter array

In this section we recall the parameter array of a Leonard system.

**Definition 5.1** [29, Theorem 4.6] Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system with eigenvalue sequence  $\{\theta_i\}_{i=0}^d$  and dual eigenvalue sequence  $\{\theta_i^*\}_{i=0}^d$ . For  $1 \leq i \leq d$  we define scalars

$$\varphi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(\tau_i(A)E_0^*)}{\text{tr}(\tau_{i-1}(A)E_0^*)}, \quad (7)$$

$$\phi_i = (\theta_0^* - \theta_i^*) \frac{\text{tr}(\eta_i(A)E_0^*)}{\text{tr}(\eta_{i-1}(A)E_0^*)}, \quad (8)$$

where  $\text{tr}$  means trace. In (7), (8) the denominators are nonzero by [29, Corollary 4.5]. Also by [29, Corollary 4.5] each of  $\varphi_i$ ,  $\phi_i$  is nonzero for  $1 \leq i \leq d$ . The sequence  $\{\varphi_i\}_{i=1}^d$  (resp.  $\{\phi_i\}_{i=1}^d$ ) is called the *first split sequence* (resp. *second split sequence*) of  $\Phi$ .

**Definition 5.2** Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system. By the *parameter array* of  $\Phi$  we mean the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \quad (9)$$

where  $\{\theta_i\}_{i=0}^d$  (resp.  $\{\theta_i^*\}_{i=0}^d$ ) denotes the eigenvalue sequence (resp. dual eigenvalue sequence) of  $\Phi$  and  $\{\varphi_i\}_{i=1}^d$  (resp.  $\{\phi_i\}_{i=1}^d$ ) denotes the first split sequence (resp. second split sequence) of  $\Phi$ . For notational convenience we abbreviate

$$\varphi := \varphi_1\varphi_2 \cdots \varphi_d, \quad \phi := \phi_1\phi_2 \cdots \phi_d.$$

The  $D_4$  action affects the parameter array as follows.

**Lemma 5.3** [41, Theorem 1.11] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Then the following (i)–(iii) hold.*

(i) *The parameter array of  $\Phi^*$  is*

$$(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d).$$

(ii) *The parameter array of  $\Phi^\downarrow$  is*

$$(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d).$$

(iii) *The parameter array of  $\Phi^\updownarrow$  is*

$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d).$$

## 6 The antiautomorphism $\dagger$

Associated with a given Leonard system in  $\mathcal{A}$ , there is a certain antiautomorphism of  $\mathcal{A}$  denoted by  $\dagger$  and defined below. Recall an *antiautomorphism* of  $\mathcal{A}$  is an isomorphism of  $\mathbb{K}$ -vector spaces  $\sigma : \mathcal{A} \rightarrow \mathcal{A}$  such that  $(XY)^\sigma = Y^\sigma X^\sigma$  for all  $X, Y \in \mathcal{A}$ .

**Lemma 6.1** [49, Theorem 6.1] *Let  $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Then there exists a unique antiautomorphism  $\dagger$  of  $\mathcal{A}$  such that  $A^\dagger = A$  and  $A^{*\dagger} = A^*$ .*

**Lemma 6.2** [49, Lemma 6.3] *Let  $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\dagger$  denote the corresponding antiautomorphism from Lemma 6.1. Then (i), (ii) hold below.*

(i) *Let  $\mathcal{D}$  denote the subalgebra of  $\mathcal{A}$  generated by  $A$ . Then  $X^\dagger = X$  for all  $X \in \mathcal{D}$ ; in particular  $E_i^\dagger = E_i$  for  $0 \leq i \leq d$ .*

(ii) *Let  $\mathcal{D}^*$  denote the subalgebra of  $\mathcal{A}$  generated by  $A^*$ . Then  $X^\dagger = X$  for all  $X \in \mathcal{D}^*$ ; in particular  $E_i^{*\dagger} = E_i^*$  for  $0 \leq i \leq d$ .*

## 7 Some reduction rules

In this section we give some formulae involving Leonard systems which we will use later in the paper.

**Lemma 7.1** *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Then for  $0 \leq i, j \leq d$  we have*

$$E_0 \tau_i^*(A^*) \tau_j(A) E_0^* = \delta_{i,j} \varphi_1 \varphi_2 \cdots \varphi_i E_0 E_0^*, \quad (10)$$

$$E_0 \eta_i^*(A^*) \tau_j(A) E_d^* = \delta_{i,j} \phi_d \phi_{d-1} \cdots \phi_{d-i+1} E_0 E_d^*, \quad (11)$$

$$E_d \tau_i^*(A^*) \eta_j(A) E_0^* = \delta_{i,j} \phi_1 \phi_2 \cdots \phi_i E_d E_0^*, \quad (12)$$

$$E_d \eta_i^*(A^*) \eta_j(A) E_d^* = \delta_{i,j} \varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1} E_d E_d^*, \quad (13)$$

$$E_0^* \tau_i(A) \tau_j^*(A^*) E_0 = \delta_{i,j} \varphi_1 \varphi_2 \cdots \varphi_i E_0^* E_0, \quad (14)$$

$$E_0^* \eta_i(A) \tau_j^*(A^*) E_d = \delta_{i,j} \phi_1 \phi_2 \cdots \phi_i E_0^* E_d, \quad (15)$$

$$E_d^* \tau_i(A) \eta_j^*(A^*) E_0 = \delta_{i,j} \phi_d \phi_{d-1} \cdots \phi_{d-i+1} E_d^* E_0, \quad (16)$$

$$E_d^* \eta_i(A) \eta_j^*(A^*) E_d = \delta_{i,j} \varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1} E_d^* E_d. \quad (17)$$

**Proof.** We first show (10). Replacing  $\Phi$  by  $\Phi^*$  and using  $\dagger$  if necessary we may assume  $i \geq j$ . In the left-hand side of (10) we insert a factor  $I$  between  $\tau_i^*(A^*)$  and  $\tau_j(A)$ . We expand using  $I = \sum_{r=0}^d E_r^*$  and simplify the result using  $\tau_i^*(A^*) E_r^* = \tau_i^*(\theta_r^*) E_r^*$  for  $0 \leq r \leq d$ . By these comments the left-hand side of (10) is equal to

$$\sum_{r=0}^d \tau_i^*(\theta_r^*) E_0 E_r^* \tau_j(A) E_0^*. \quad (18)$$

For  $0 \leq r \leq d$  we examine term  $r$  in (18). By Definition 3.2 we have  $\tau_i^*(\theta_r^*) = 0$  if  $r < i$ . By [49, Lemma 5.10(i)]  $E_r^* A^s E_0^* = 0$  for  $0 \leq s \leq r-1$ . By this and since the polynomial  $\tau_j$  is monic of degree  $j$ ,  $E_r^* \tau_j(A) E_0^*$  is 0 if  $j < r$  and  $E_r^* A^r E_0^*$  if  $j = r$ . Referring to [49, Theorem 8.4] we have  $E_i^* A^i E_0^* = p_i(A) E_0^*$ . Note that  $E_0 p_i(A) = p_i(\theta_0) E_0$  by construction and  $\tau_i^*(\theta_i^*) p_i(\theta_0) = \varphi_1 \varphi_2 \cdots \varphi_i$  by [49, Lemma 17.5]. By these comments and since  $i \geq j$  the sum (18) is equal to the right-hand side of (10). We have now verified (10). To get (11)–(17), apply  $D_4$  to (10) and use Lemma 5.3.  $\square$

**Lemma 7.2** Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Then for  $0 \leq r \leq d$  we have

$$E_0^* E_d E_d^* E_r = \frac{\varphi}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)} \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_0^* E_r, \quad (19)$$

$$E_0^* E_0 E_d^* E_r = \frac{\varphi}{\eta_d(\theta_0) \tau_d^*(\theta_d^*)} \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_0^* E_r, \quad (20)$$

$$E_d^* E_d E_0^* E_r = \frac{\phi}{\tau_d(\theta_d) \eta_d^*(\theta_0^*)} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} E_d^* E_r, \quad (21)$$

$$E_d^* E_0 E_0^* E_r = \frac{\phi}{\eta_d(\theta_0) \eta_d^*(\theta_0^*)} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} E_d^* E_r, \quad (22)$$

$$E_0 E_d^* E_d E_r^* = \frac{\varphi}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)} \frac{\phi_1 \phi_2 \cdots \phi_r}{\varphi_1 \varphi_2 \cdots \varphi_r} E_0 E_r^*, \quad (23)$$

$$E_0 E_0^* E_d E_r^* = \frac{\varphi}{\tau_d(\theta_d) \eta_d^*(\theta_0^*)} \frac{\phi_1 \phi_2 \cdots \phi_r}{\varphi_1 \varphi_2 \cdots \varphi_r} E_0 E_r^*, \quad (24)$$

$$E_d E_d^* E_0 E_r^* = \frac{\phi}{\eta_d(\theta_0) \tau_d^*(\theta_d^*)} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_1 \phi_2 \cdots \phi_r} E_d E_r^*, \quad (25)$$

$$E_d E_0^* E_0 E_r^* = \frac{\phi}{\eta_d(\theta_0) \eta_d^*(\theta_0^*)} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_1 \phi_2 \cdots \phi_r} E_d E_r^*. \quad (26)$$

**Proof.** We first show (19). Following [33, Definition 5.1] we define

$$S = \sum_{i=0}^d \frac{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}{\varphi_1 \varphi_2 \cdots \varphi_i} E_i. \quad (27)$$

By [33, Lemma 11.5],

$$S E_0^* = \frac{\tau_d(\theta_d) \tau_d^*(\theta_d^*)}{\varphi} E_d^* E_d E_0^*. \quad (28)$$

Applying  $\dagger$  to (28) using Lemma 6.2, and multiplying the result on the right by  $E_r$  we find

$$E_0^* S E_r = \frac{\tau_d(\theta_d) \tau_d^*(\theta_d^*)}{\varphi} E_0^* E_d E_d^* E_r. \quad (29)$$

Using (27) we find

$$S E_r = \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r. \quad (30)$$

Combining (29) and (30) we obtain (19). To get (20)–(26), apply  $D_4$  to (19) and use Lemma 5.3.  $\square$

## 8 Some traces

Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system. In our main results we will need the scalars

$$\text{tr}(E_r E_0^*), \quad \text{tr}(E_r E_d^*), \quad \text{tr}(E_r^* E_0), \quad \text{tr}(E_r^* E_d)$$

for  $0 \leq r \leq d$ . In this section we make some comments about these scalars.

**Lemma 8.1** [49, Lemma 9.2] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system. Then*

$$E_r E_0^* E_r = \text{tr}(E_r E_0^*) E_r, \quad E_r E_d^* E_r = \text{tr}(E_r E_d^*) E_r, \quad (31)$$

$$E_r^* E_0 E_r^* = \text{tr}(E_r^* E_0) E_r^*, \quad E_r^* E_d E_r^* = \text{tr}(E_r^* E_d) E_r^*, \quad (32)$$

$$E_0^* E_r E_0^* = \text{tr}(E_r E_0^*) E_0^*, \quad E_d^* E_r E_d^* = \text{tr}(E_r E_d^*) E_d^*, \quad (33)$$

$$E_0 E_r^* E_0 = \text{tr}(E_r^* E_0) E_0, \quad E_d E_r^* E_d = \text{tr}(E_r^* E_d) E_d. \quad (34)$$

**Lemma 8.2** [49, Theorem 17.12] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Then for  $0 \leq r \leq d$  we have*

$$\text{tr}(E_r E_0^*) = \frac{\varphi_1 \varphi_2 \cdots \varphi_r \phi_1 \phi_2 \cdots \phi_{d-r}}{\eta_d^*(\theta_0^*) \tau_r(\theta_r) \eta_{d-r}(\theta_r)}, \quad (35)$$

$$\text{tr}(E_r E_d^*) = \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1} \varphi_d \varphi_{d-1} \cdots \varphi_{r+1}}{\tau_d^*(\theta_d^*) \tau_r(\theta_r) \eta_{d-r}(\theta_r)}, \quad (36)$$

$$\text{tr}(E_r^* E_0) = \frac{\varphi_1 \varphi_2 \cdots \varphi_r \phi_d \phi_{d-1} \cdots \phi_{r+1}}{\eta_d(\theta_0) \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*)}, \quad (37)$$

$$\text{tr}(E_r^* E_d) = \frac{\phi_1 \phi_2 \cdots \phi_r \varphi_d \varphi_{d-1} \cdots \varphi_{r+1}}{\tau_d(\theta_d) \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*)}. \quad (38)$$

**Corollary 8.3** *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Then for  $0 \leq r \leq d$  we have*

$$\text{tr}(E_0 E_0^*) = \frac{\phi}{\eta_d(\theta_0) \eta_d^*(\theta_0^*)}, \quad \text{tr}(E_0 E_d^*) = \frac{\varphi}{\eta_d(\theta_0) \tau_d^*(\theta_d^*)}, \quad (39)$$

$$\text{tr}(E_d E_0^*) = \frac{\varphi}{\tau_d(\theta_d) \eta_d^*(\theta_0^*)}, \quad \text{tr}(E_d E_d^*) = \frac{\phi}{\tau_d(\theta_d) \tau_d^*(\theta_d^*)}. \quad (40)$$

**Proof.** Follows from Lemma 8.2.  $\square$

**Corollary 8.4** *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system. Then each of  $\text{tr}(E_r E_0^*)$ ,  $\text{tr}(E_r E_d^*)$ ,  $\text{tr}(E_r^* E_0)$ ,  $\text{tr}(E_r^* E_d)$  is nonzero for  $0 \leq r \leq d$ .*

**Proof.** By Lemma 8.2 and since each of  $\varphi_i, \phi_i$  is nonzero for  $1 \leq i \leq d$ .  $\square$

## 9 A bilinear form

In this section we associate with each Leonard system a certain bilinear form. To prepare for this we recall a few concepts from linear algebra.

By a *bilinear form on  $V$*  we mean a map  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$  that satisfies the following four conditions for all  $u, v, w \in V$  and for all  $\alpha \in \mathbb{K}$ : (i)  $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ ; (ii)  $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ ; (iii)  $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ ; (iv)  $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$ . Let  $\langle \cdot, \cdot \rangle$  denote a bilinear form on  $V$ . This form is said to be *symmetric* whenever  $\langle u, v \rangle = \langle v, u \rangle$  for all

$u, v \in V$ . Let  $\langle \cdot, \cdot \rangle$  denote a bilinear form on  $V$ . Then the following are equivalent: (i) there exists a nonzero  $u \in V$  such that  $\langle u, v \rangle = 0$  for all  $v \in V$ ; (ii) there exists a nonzero  $v \in V$  such that  $\langle u, v \rangle = 0$  for all  $u \in V$ . The form  $\langle \cdot, \cdot \rangle$  is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. Let  $\sigma : \mathcal{A} \rightarrow \mathcal{A}$  denote an antiautomorphism. Then there exists a nonzero bilinear form  $\langle \cdot, \cdot \rangle$  on  $V$  such that  $\langle Xu, v \rangle = \langle u, X^\sigma v \rangle$  for all  $u, v \in V$  and for all  $X \in \mathcal{A}$ . The form is unique up to multiplication by a nonzero scalar in  $\mathbb{K}$ . The form is nondegenerate. We refer to this form as the *bilinear form on  $V$  associated with  $\sigma$* . This form is not symmetric in general.

We now return our attention to Leonard systems.

**Definition 9.1** Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Let  $\dagger : \mathcal{A} \rightarrow \mathcal{A}$  denote the corresponding antiautomorphism from Lemma 6.1. For the rest of this paper we let  $\langle \cdot, \cdot \rangle$  denote the bilinear form on  $V$  associated with  $\dagger$ . By construction, for  $X \in \mathcal{A}$  we have

$$\langle Xu, v \rangle = \langle u, X^\dagger v \rangle \quad (u, v \in V). \quad (41)$$

**Lemma 9.2** [49, Lemma 15.2] Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Let  $\mathcal{D}$  (resp.  $\mathcal{D}^*$ ) denote the subalgebra of  $\mathcal{A}$  generated by  $A$  (resp.  $A^*$ ). Then for  $X \in \mathcal{D} \cup \mathcal{D}^*$  we have

$$\langle Xu, v \rangle = \langle u, Xv \rangle \quad (u, v \in V). \quad (42)$$

**Lemma 9.3** [49, Corollary 15.4] Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . Then the bilinear form  $\langle \cdot, \cdot \rangle$  is symmetric.

Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$ . In our discussion of  $\Phi$  we will refer to the inner products

$$\begin{array}{cccc} \langle \xi_0, \xi_0^* \rangle, & \langle \xi_0, \xi_d^* \rangle, & \langle \xi_d, \xi_0^* \rangle, & \langle \xi_d, \xi_d^* \rangle, \\ \langle \xi_0, \xi_0 \rangle, & \langle \xi_d, \xi_d \rangle, & \langle \xi_0^*, \xi_0^* \rangle, & \langle \xi_d^*, \xi_d^* \rangle, \end{array} \quad (43)$$

where the vectors  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  are from Lemma 3.3. We have some comments on these inner products.

**Lemma 9.4** [49, Lemma 15.5] Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  denote nonzero vectors in  $V$  that satisfy (4). Then each of the scalars (43) is nonzero.

**Lemma 9.5** [49, Lemma 15.5] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  denote nonzero vectors in  $V$  that satisfy (4). Then*

$$E_0 \xi_0^* = \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_0, \xi_0 \rangle} \xi_0, \quad E_d \xi_0^* = \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_d, \xi_d \rangle} \xi_d, \quad (44)$$

$$E_0 \xi_d^* = \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_0, \xi_0 \rangle} \xi_0, \quad E_d \xi_d^* = \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d, \xi_d \rangle} \xi_d, \quad (45)$$

$$E_0^* \xi_0 = \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_0^*, \xi_0^* \rangle} \xi_0^*, \quad E_d^* \xi_0 = \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_d^*, \xi_d^* \rangle} \xi_d^*, \quad (46)$$

$$E_0^* \xi_d = \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_0^*, \xi_0^* \rangle} \xi_0^*, \quad E_d^* \xi_d = \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d^*, \xi_d^* \rangle} \xi_d^*. \quad (47)$$

In the following two lemmas we give some relationships involving the eight scalars in (43).

**Lemma 9.6** [49, Lemma 15.5] *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  denote nonzero vectors in  $V$  that satisfy (4). Then*

$$\langle \xi_0, \xi_0^* \rangle^2 = \text{tr}(E_0 E_0^*) \langle \xi_0, \xi_0 \rangle \langle \xi_0^*, \xi_0^* \rangle, \quad (48)$$

$$\langle \xi_0, \xi_d^* \rangle^2 = \text{tr}(E_0 E_d^*) \langle \xi_0, \xi_0 \rangle \langle \xi_d^*, \xi_d^* \rangle, \quad (49)$$

$$\langle \xi_d, \xi_0^* \rangle^2 = \text{tr}(E_d E_0^*) \langle \xi_d, \xi_d \rangle \langle \xi_0^*, \xi_0^* \rangle, \quad (50)$$

$$\langle \xi_d, \xi_d^* \rangle^2 = \text{tr}(E_d E_d^*) \langle \xi_d, \xi_d \rangle \langle \xi_d^*, \xi_d^* \rangle. \quad (51)$$

**Lemma 9.7** *Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  denote nonzero vectors in  $V$  that satisfy (4). Then*

$$\frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_0, \xi_d^* \rangle} = \frac{\phi \langle \xi_d, \xi_0^* \rangle}{\varphi \langle \xi_d, \xi_d^* \rangle}. \quad (52)$$

**Proof.** Let  $S$  be from (27). Recall  $\xi_0 \in E_0 V$  so  $S\xi_0 = \xi_0$ . Similarly  $\xi_d \in E_d V$  so  $S\xi_d = \phi\varphi^{-1}\xi_d$ . By [33, Theorem 6.7] we have  $SE_0^* V = E_d^* V$  so there exists a nonzero  $t \in \mathbb{K}$  such that  $S\xi_0^* = t\xi_d^*$ . We may now argue

$$\langle \xi_d, \xi_d^* \rangle = t^{-1} \langle \xi_d, S\xi_0^* \rangle = t^{-1} \langle S\xi_d, \xi_0^* \rangle = \phi\varphi^{-1} t^{-1} \langle \xi_d, \xi_0^* \rangle$$

and

$$\langle \xi_0, \xi_d^* \rangle = t^{-1} \langle \xi_0, S\xi_0^* \rangle = t^{-1} \langle S\xi_0, \xi_0^* \rangle = t^{-1} \langle \xi_0, \xi_0^* \rangle.$$

Using these comments line (52) is routinely verified.  $\square$

## 10 Transition maps from $\{E_i \xi_0^*\}_{i=0}^d$ and $\{E_{d-i} \xi_0^*\}_{i=0}^d$

In Lemma 3.3 we gave 24 bases for  $V$ . We are now ready to display the transition maps between ordered pairs of bases among these 24. We start with the transition maps from the basis  $\{E_i \xi_0^*\}_{i=0}^d$  and the basis  $\{E_{d-i} \xi_0^*\}_{i=0}^d$ .

**Notation 10.1** Let  $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$  denote a Leonard system in  $\mathcal{A}$  and let  $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$  denote the parameter array of  $\Phi$ . Let  $\xi_0, \xi_d, \xi_0^*, \xi_d^*$  denote nonzero vectors in  $V$  that satisfy (4).

**Theorem 10.2** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\sum_{r=0}^d \frac{X_r E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_i \xi_0^* = X_i \xi_0^*, \quad (53)$$

$$\frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \tau_r(\theta_r) \eta_{d-r}(\theta_r) X_r E_d^* E_r \cdot E_i \xi_0^* = X_i \xi_d^*, \quad (54)$$

$$\sum_{r=0}^d \frac{X_{d-r} E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_{d-i} \xi_0^* = X_i \xi_0^*, \quad (55)$$

$$\frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \tau_r(\theta_r) \eta_{d-r}(\theta_r) X_{d-r} E_d^* E_r \cdot E_{d-i} \xi_0^* = X_i \xi_d^*. \quad (56)$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_0 E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_i \xi_0^* = X_i \xi_0, \quad (57)$$

$$\frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_d E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_i \xi_0^* = X_i \xi_d, \quad (58)$$

$$\frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0 E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_{d-i} \xi_0^* = X_i \xi_0, \quad (59)$$

$$\frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_{d-i} \xi_0^* = X_i \xi_d. \quad (60)$$

**Proof.** For  $0 \leq i, r \leq d$  we have  $E_r E_i = \delta_{r,i} E_r$ . From this and the equation on the left in (33),

$$\frac{E_0^* E_r}{\text{tr}(E_r E_0^*)} \cdot E_i E_0^* = \delta_{r,i} E_0^* \quad (0 \leq r, i \leq d). \quad (61)$$

(i): To get (53), multiply each side of (61) on the left by  $X_r$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$ . To get (54), multiply each side of (61) on the left by  $X_r E_d^* E_d$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$ , the equations on the right in (44), (47), equations (35), (51), the equation on the right in (40), and (21). To get (55), in line (53) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ . To get (56), in line (54) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ .

(ii): To get (57), multiply each side of (61) on the left by  $X_r E_0$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$  and the equation on the left in (44). To get (58), multiply each side of (61) on the left by  $X_r E_d$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$  and the equation on the right in (44). To get (59), in line (57) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ . To get (60), in line (58) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ .  $\square$

## 11 Transition maps from $\{E_i \xi_d^*\}_{i=0}^d$ and $\{E_{d-i} \xi_d^*\}_{i=0}^d$

**Theorem 11.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \tau_r(\theta_r) \eta_{d-r}(\theta_r) X_r E_0^* E_r \cdot E_i \xi_d^* &= X_i \xi_0^*, \\ \sum_{r=0}^d \frac{X_r E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_i \xi_d^* &= X_i \xi_d^*, \\ \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \tau_r(\theta_r) \eta_{d-r}(\theta_r) X_{d-r} E_0^* E_r \cdot E_{d-i} \xi_d^* &= X_i \xi_0^*, \\ \sum_{r=0}^d \frac{X_{d-r} E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_{d-i} \xi_d^* &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_0 E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_i \xi_d^* &= X_i \xi_0, \\ \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_d E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_i \xi_d^* &= X_i \xi_d, \\ \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0 E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_{d-i} \xi_d^* &= X_i \xi_0, \\ \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d E_d^* E_r}{\text{tr}(E_r E_d^*)} \cdot E_{d-i} \xi_d^* &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 10.2 to  $\Phi^\downarrow$ . □

## 12 Transition maps from $\{E_i^* \xi_0\}_{i=0}^d$ and $\{E_{d-i}^* \xi_0\}_{i=0}^d$

**Theorem 12.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_0^* E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_i^* \xi_0 &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_d^* E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_i^* \xi_0 &= X_i \xi_d^*, \\ \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0^* E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_{d-i}^* \xi_0 &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d^* E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_{d-i}^* \xi_0 &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \sum_{r=0}^d \frac{X_r E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_i^* \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*) X_r E_d E_r^* \cdot E_i^* \xi_0 &= X_i \xi_d, \\ \sum_{r=0}^d \frac{X_{d-r} E_0 E_r^*}{\text{tr}(E_r^* E_0)} \cdot E_{d-i}^* \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*) X_{d-r} E_d E_r^* \cdot E_{d-i}^* \xi_0 &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 10.2 to  $\Phi^*$ . □

### 13 Transition maps from $\{E_i^* \xi_d\}_{i=0}^d$ and $\{E_{d-i}^* \xi_d\}_{i=0}^d$

**Theorem 13.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_0^* E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_i^* \xi_d &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_d^* E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_i^* \xi_d &= X_i \xi_d^*, \\ \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0^* E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_{d-i}^* \xi_d &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d^* E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_{d-i}^* \xi_d &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*) X_r E_0 E_r^* \cdot E_i^* \xi_d &= X_i \xi_0, \\ \sum_{r=0}^d \frac{X_r E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_i^* \xi_d &= X_i \xi_d, \\ \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \tau_r^*(\theta_r^*) \eta_{d-r}^*(\theta_r^*) X_{d-r} E_0 E_r^* \cdot E_{d-i}^* \xi_d &= X_i \xi_0, \\ \sum_{r=0}^d \frac{X_{d-r} E_d E_r^*}{\text{tr}(E_r^* E_d)} \cdot E_{d-i}^* \xi_d &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 12.1 to  $\Phi^\downarrow$ . □

## 14 Transition maps from $\{\tau_i(A)\xi_0^*\}_{i=0}^d$ and $\{\tau_{d-i}(A)\xi_0^*\}_{i=0}^d$

**Theorem 14.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\frac{1}{\text{tr}(E_0 E_0^*)} \sum_{r=0}^d \frac{X_r E_0^* E_0 \tau_r^*(A^*)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_i(A) \xi_0^* = X_i \xi_0^*, \quad (62)$$

$$\frac{\tau_d^*(\theta_d^*)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_r E_d^* \eta_{d-r}(A) \cdot \tau_i(A) \xi_0^* = X_i \xi_d^*, \quad (63)$$

$$\frac{1}{\text{tr}(E_0 E_0^*)} \sum_{r=0}^d \frac{X_{d-r} E_0^* E_0 \tau_r^*(A^*)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_{d-i}(A) \xi_0^* = X_i \xi_0^*, \quad (64)$$

$$\frac{\tau_d^*(\theta_d^*)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_d^* \eta_{d-r}(A) \cdot \tau_{d-i}(A) \xi_0^* = X_i \xi_d^*. \quad (65)$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_0 \tau_r^*(A^*)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_i(A) \xi_0^* = X_i \xi_0, \quad (66)$$

$$\frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_r E_d E_d^* \eta_{d-r}(A) \cdot \tau_i(A) \xi_0^* = X_i \xi_d, \quad (67)$$

$$\frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0 \tau_r^*(A^*)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_{d-i}(A) \xi_0^* = X_i \xi_0, \quad (68)$$

$$\frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_d E_d^* \eta_{d-r}(A) \cdot \tau_{d-i}(A) \xi_0^* = X_i \xi_d. \quad (69)$$

**Proof.** Multiply both sides of (10) on the left by  $E_0^*$ , use the equation on the left in (33), and replace  $(i, j)$  with  $(r, i)$  to obtain

$$\frac{1}{\text{tr}(E_0 E_0^*)} \frac{E_0^* E_0 \tau_r^*(A^*)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_i(A) E_0^* = \delta_{r,i} E_0^* \quad (0 \leq r, i \leq d). \quad (70)$$

(i): To get (62), multiply each side of (70) on the left by  $X_r$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$ . Next we show (63). By [31, Corollary 5.3] we have

$$E_d^* E_0 \tau_r^*(A^*) = \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\eta_d(\theta_0)} E_d^* \eta_{d-r}(A) \quad (71)$$

for  $0 \leq r \leq d$ . Now to get (63), multiply each side of (70) on the left by  $X_r E_d^* E_0$  and on the right by  $\xi_0^*$ , sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$ , the equations on the left in (34), (44), the equation on the right in (46), equations (49), (71), and the equation on the right in (39). To get (64), in line (62) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ . To get (65), in line (63) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ .

(ii): To get (66), multiply each side of (70) on the left by  $X_r E_0$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$  and the equations on the left in (34), (44). To get (67), multiply each side of (70) on the left by  $X_r E_d E_d^* E_0$  and on the right by  $\xi_0^*$ ; now sum the resulting equation over  $r = 0, \dots, d$  and simplify using  $E_0^* \xi_0^* = \xi_0^*$ , the equations on the left in (34), (44), the equations on the right in (45), (46), equations (49), (52), (71), and the equation on the right in (39). To get (68), in line (66) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ . To get (69), in line (67) replace  $i$  by  $d - i$  and  $X_0, \dots, X_d$  by  $X_d, \dots, X_0$ .  $\square$

## 15 Transition maps from $\{\eta_i(A)\xi_0^*\}_{i=0}^d$ and $\{\eta_{d-i}(A)\xi_0^*\}_{i=0}^d$

**Theorem 15.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{1}{\text{tr}(E_d E_0^*)} \sum_{r=0}^d \frac{X_r E_0^* E_d \tau_r^*(A^*)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \eta_i(A) \xi_0^* &= X_i \xi_0^*, \\ \frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_r E_d^* \tau_{d-r}(A) \cdot \eta_i(A) \xi_0^* &= X_i \xi_d^*, \\ \frac{1}{\text{tr}(E_d E_0^*)} \sum_{r=0}^d \frac{X_{d-r} E_0^* E_d \tau_r^*(A^*)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \eta_{d-i}(A) \xi_0^* &= X_i \xi_0^*, \\ \frac{\tau_d^*(\theta_d^*)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_d^* \tau_{d-r}(A) \cdot \eta_{d-i}(A) \xi_0^* &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\tau_d^*(\theta_d^*)}{\varphi} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_r E_0 E_d^* \tau_{d-r}(A) \cdot \eta_i(A) \xi_0^* &= X_i \xi_0, \\ \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_d \tau_r^*(A^*)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \eta_i(A) \xi_0^* &= X_i \xi_d, \\ \frac{\tau_d^*(\theta_d^*)}{\varphi} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_0 E_d^* \tau_{d-r}(A) \cdot \eta_{d-i}(A) \xi_0^* &= X_i \xi_0, \\ \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d \tau_r^*(A^*)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \eta_{d-i}(A) \xi_0^* &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 14.1 to  $\Phi^\downarrow$ . □

## 16 Transition maps from $\{\tau_i(A)\xi_d^*\}_{i=0}^d$ and $\{\tau_{d-i}(A)\xi_d^*\}_{i=0}^d$

**Theorem 16.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\eta_d^*(\theta_0^*)}{\phi} \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_r E_0^* \eta_{d-r}(A) \cdot \tau_i(A) \xi_d^* &= X_i \xi_0^*, \\ \frac{1}{\text{tr}(E_0 E_d^*)} \sum_{r=0}^d \frac{X_r E_d^* E_0 \eta_r^*(A^*)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \tau_i(A) \xi_d^* &= X_i \xi_d^*, \\ \frac{\eta_d^*(\theta_0^*)}{\phi} \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_0^* \eta_{d-r}(A) \cdot \tau_{d-i}(A) \xi_d^* &= X_i \xi_0^*, \\ \frac{1}{\text{tr}(E_0 E_d^*)} \sum_{r=0}^d \frac{X_{d-r} E_d^* E_0 \eta_r^*(A^*)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \tau_{d-i}(A) \xi_d^* &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_0 \eta_r^*(A^*)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \tau_i(A) \xi_d^* &= X_i \xi_0, \\ \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_r E_d E_0^* \eta_{d-r}(A) \cdot \tau_i(A) \xi_d^* &= X_i \xi_d, \\ \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0 \eta_r^*(A^*)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \tau_{d-i}(A) \xi_d^* &= X_i \xi_0, \\ \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_d E_0^* \eta_{d-r}(A) \cdot \tau_{d-i}(A) \xi_d^* &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 14.1 to  $\Phi^\downarrow$ . □

## 17 Transition maps from $\{\eta_i(A)\xi_d^*\}_{i=0}^d$ and $\{\eta_{d-i}(A)\xi_d^*\}_{i=0}^d$

**Theorem 17.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} & \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_r E_0^* \tau_{d-r}(A) \cdot \eta_i(A) \xi_d^* = X_i \xi_0^*, \\ & \frac{1}{\text{tr}(E_d E_d^*)} \sum_{r=0}^d \frac{X_r E_d^* E_d \eta_r^*(A^*)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_i(A) \xi_d^* = X_i \xi_d^*, \\ & \frac{\eta_d^*(\theta_0^*)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_0^* \tau_{d-r}(A) \cdot \eta_{d-i}(A) \xi_d^* = X_i \xi_0^*, \\ & \frac{1}{\text{tr}(E_d E_d^*)} \sum_{r=0}^d \frac{X_{d-r} E_d^* E_d \eta_r^*(A^*)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_{d-i}(A) \xi_d^* = X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} & \frac{\eta_d^*(\theta_0^*)}{\phi} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_r E_0 E_0^* \tau_{d-r}(A) \cdot \eta_i(A) \xi_d^* = X_i \xi_0, \\ & \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_d \eta_r^*(A^*)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_i(A) \xi_d^* = X_i \xi_d, \\ & \frac{\eta_d^*(\theta_0^*)}{\phi} \frac{\langle \xi_0, \xi_0 \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_0 E_0^* \tau_{d-r}(A) \cdot \eta_{d-i}(A) \xi_d^* = X_i \xi_0, \\ & \frac{\langle \xi_d, \xi_d \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d \eta_r^*(A^*)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_{d-i}(A) \xi_d^* = X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 14.1 to  $\Phi^{\downarrow\downarrow}$ . □

## 18 Transition maps from $\{\tau_i^*(A^*)\xi_0\}_{i=0}^d$ and $\{\tau_{d-i}^*(A^*)\xi_0\}_{i=0}^d$

**Theorem 18.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_0^* \tau_r(A)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_i^*(A^*) \xi_0 &= X_i \xi_0^*, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_r E_d^* E_d \eta_{d-r}^*(A^*) \cdot \tau_i^*(A^*) \xi_0 &= X_i \xi_d^*, \\ \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0^* \tau_r(A)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_{d-i}^*(A^*) \xi_0 &= X_i \xi_0^*, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_d^* E_d \eta_{d-r}^*(A^*) \cdot \tau_{d-i}^*(A^*) \xi_0 &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{1}{\text{tr}(E_0 E_0^*)} \sum_{r=0}^d \frac{X_r E_0 E_0^* \tau_r(A)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_i^*(A^*) \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_r E_d \eta_{d-r}^*(A^*) \cdot \tau_i^*(A^*) \xi_0 &= X_i \xi_d, \\ \frac{1}{\text{tr}(E_0 E_0^*)} \sum_{r=0}^d \frac{X_{d-r} E_0 E_0^* \tau_r(A)}{\varphi_1 \varphi_2 \cdots \varphi_r} \cdot \tau_{d-i}^*(A^*) \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\varphi} \frac{\langle \xi_d, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_d \eta_{d-r}^*(A^*) \cdot \tau_{d-i}^*(A^*) \xi_0 &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 14.1 to  $\Phi^*$ . □

## 19 Transition maps from $\{\eta_i^*(A^*)\xi_0\}_{i=0}^d$ and $\{\eta_{d-i}^*(A^*)\xi_0\}_{i=0}^d$

**Theorem 19.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\tau_d(\theta_d)}{\varphi} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_r E_0^* E_d \tau_{d-r}^*(A^*) \cdot \eta_i^*(A^*) \xi_0 &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_d^* \tau_r(A)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \eta_i^*(A^*) \xi_0 &= X_i \xi_d^*, \\ \frac{\tau_d(\theta_d)}{\varphi} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_0^* E_d \tau_{d-r}^*(A^*) \cdot \eta_{d-i}^*(A^*) \xi_0 &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_0, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d^* \tau_r(A)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \eta_{d-i}^*(A^*) \xi_0 &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{1}{\text{tr}(E_0 E_d^*)} \sum_{r=0}^d \frac{X_r E_0 E_d^* \tau_r(A)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \eta_i^*(A^*) \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_r E_d \tau_{d-r}^*(A^*) \cdot \eta_i^*(A^*) \xi_0 &= X_i \xi_d, \\ \frac{1}{\text{tr}(E_0 E_d^*)} \sum_{r=0}^d \frac{X_{d-r} E_0 E_d^* \tau_r(A)}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} \cdot \eta_{d-i}^*(A^*) \xi_0 &= X_i \xi_0, \\ \frac{\tau_d(\theta_d)}{\phi} \frac{\langle \xi_d, \xi_d^* \rangle}{\langle \xi_0, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_d \tau_{d-r}^*(A^*) \cdot \eta_{d-i}^*(A^*) \xi_0 &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 18.1 to  $\Phi^\downarrow$ . □

## 20 Transition maps from $\{\tau_i^*(A^*)\xi_d\}_{i=0}^d$ and $\{\tau_{d-i}^*(A^*)\xi_d\}_{i=0}^d$

**Theorem 20.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} & \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_r E_0^* \eta_r(A)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \tau_i^*(A^*) \xi_d = X_i \xi_0^*, \\ & \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_r E_d^* E_0 \eta_{d-r}^*(A^*) \cdot \tau_i^*(A^*) \xi_d = X_i \xi_d^*, \\ & \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_0^* \eta_r(A)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \tau_{d-i}^*(A^*) \xi_d = X_i \xi_0^*, \\ & \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_d^* E_0 \eta_{d-r}^*(A^*) \cdot \tau_{d-i}^*(A^*) \xi_d = X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} & \frac{\eta_d(\theta_0)}{\phi} \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_r E_0 \eta_{d-r}^*(A^*) \cdot \tau_i^*(A^*) \xi_d = X_i \xi_0, \\ & \frac{1}{\text{tr}(E_d E_0^*)} \sum_{r=0}^d \frac{X_r E_d E_0^* \eta_r(A)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \tau_i^*(A^*) \xi_d = X_i \xi_d, \\ & \frac{\eta_d(\theta_0)}{\phi} \frac{\langle \xi_0, \xi_0^* \rangle}{\langle \xi_d, \xi_0^* \rangle} \sum_{r=0}^d X_{d-r} E_0 \eta_{d-r}^*(A^*) \cdot \tau_{d-i}^*(A^*) \xi_d = X_i \xi_0, \\ & \frac{1}{\text{tr}(E_d E_0^*)} \sum_{r=0}^d \frac{X_{d-r} E_d E_0^* \eta_r(A)}{\phi_1 \phi_2 \cdots \phi_r} \cdot \tau_{d-i}^*(A^*) \xi_d = X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 18.1 to  $\Phi^\downarrow$ . □

## 21 Transition maps from $\{\eta_i^*(A^*)\xi_d\}_{i=0}^d$ and $\{\eta_{d-i}^*(A^*)\xi_d\}_{i=0}^d$

**Theorem 21.1** Referring to Notation 10.1 the following (i), (ii) hold.

(i) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i\}_{i=0}^d; \quad \{E_{d-i}\}_{i=0}^d; \quad \{\tau_i(A)\}_{i=0}^d; \quad \{\tau_{d-i}(A)\}_{i=0}^d; \quad \{\eta_i(A)\}_{i=0}^d; \quad \{\eta_{d-i}(A)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\eta_d(\theta_0)}{\phi} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_r E_0^* E_0 \tau_{d-r}^*(A^*) \cdot \eta_i^*(A^*) \xi_d &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_r E_d^* \eta_r(A)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_i^*(A^*) \xi_d &= X_i \xi_d^*, \\ \frac{\eta_d(\theta_0)}{\phi} \frac{\langle \xi_0^*, \xi_0^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_0^* E_0 \tau_{d-r}^*(A^*) \cdot \eta_{d-i}^*(A^*) \xi_d &= X_i \xi_0^*, \\ \frac{\langle \xi_d^*, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d \frac{X_{d-r} E_d^* \eta_r(A)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_{d-i}^*(A^*) \xi_d &= X_i \xi_d^*. \end{aligned}$$

(ii) Let  $\{X_i\}_{i=0}^d$  denote one of

$$\{E_i^*\}_{i=0}^d; \quad \{E_{d-i}^*\}_{i=0}^d; \quad \{\tau_i^*(A^*)\}_{i=0}^d; \quad \{\tau_{d-i}^*(A^*)\}_{i=0}^d; \quad \{\eta_i^*(A^*)\}_{i=0}^d; \quad \{\eta_{d-i}^*(A^*)\}_{i=0}^d.$$

Then for  $0 \leq i \leq d$ ,

$$\begin{aligned} \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_r E_0 \tau_{d-r}^*(A^*) \cdot \eta_i^*(A^*) \xi_d &= X_i \xi_0, \\ \frac{1}{\text{tr}(E_d E_d^*)} \sum_{r=0}^d \frac{X_r E_d E_d^* \eta_r(A)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_i^*(A^*) \xi_d &= X_i \xi_d, \\ \frac{\eta_d(\theta_0)}{\varphi} \frac{\langle \xi_0, \xi_d^* \rangle}{\langle \xi_d, \xi_d^* \rangle} \sum_{r=0}^d X_{d-r} E_0 \tau_{d-r}^*(A^*) \cdot \eta_{d-i}^*(A^*) \xi_d &= X_i \xi_0, \\ \frac{1}{\text{tr}(E_d E_d^*)} \sum_{r=0}^d \frac{X_{d-r} E_d E_d^* \eta_r(A)}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-r+1}} \cdot \eta_{d-i}^*(A^*) \xi_d &= X_i \xi_d. \end{aligned}$$

**Proof.** Apply Theorem 18.1 to  $\Phi^{\downarrow\downarrow}$ . □

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